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Non-commutative harmonic oscillators and Fuchsian ordinary differential operators

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Abstract. The spectral problem for non-commutative harmonic oscillators is shown to be equivalent to solve Fuchsian ordinary differential equations with four regular points in a complex domain.

1. Introduction

We study the spectral problem of the (2×2) matrix-valued (ordinary) differential operator

$$A(-\frac{\partial_x^2}{2}+\frac{x^2}{2})+B(x\partial_x+\frac{1}{2})=\begin{bmatrix} a_{11}(-\frac{\partial_x^2}{2}+\frac{x^2}{2}) & a_{12}(-\frac{\partial_x^2}{2}+\frac{x^2}{2})-b(x\partial_x+\frac{1}{2}) \\ a_{12}(-\frac{\partial_x^2}{2}+\frac{x^2}{2})+b(x\partial_x+\frac{1}{2}) & a_{22}(-\frac{\partial_x^2}{2}+\frac{x^2}{2}) \end{bmatrix},$$

where $A, B \in \mathrm{Mat}_2(\mathbf{R})$, $A={}^tA$ is positive definite, and $B=-{}^tB$. An eigenfunction $u \in L^2(\mathbf{R}^2, \mathbf{C}^2)$ is a $\mathbf{C}^2$-valued L^2-function on $\mathbf{R}$, and $\mu \in \mathbf{C}$ represents the spectrum. The eigenvalue problem for this operator was first introduced and studied by Parmeggiani and Wakayama [PW]. We assume that $\det A > \det B > 0$. In such a case, as is seen below, the eigenvalues are all positive and form a discrete set with finite multiplicity.

The operator above possesses two kinds of non-commutativity, non-commutativity with respect to multiplication of the matrices and that with respect to differential operators. The interaction of these two results in a non-trivial relation to the 'connection problem' concerning an ordinary differential operator in a complex domain.

Theorem 1. *The eigenvalue problem*

$$\left[A(-\frac{\partial_x^2}{2}+\frac{x^2}{2})+B(x\partial_x+\frac{1}{2})-\mu I\right]u(x)=0 \tag{1.1}$$

is equivalent to the problem

$$P(z,D_z)U(z)=0 \tag{1.2}$$

for holomorphic functions $U(z)$ on the unit disk.

Here, $P(z,D_z)$ is a third-order linear ordinary differential operator defined in terms of A, B and μ. The definition of $P(z,D_z)$ is given in Section 3.2.

We note that every solution $u(x)$ of the equation (1.1) is the sum of an even function solution $u^e(x)$ and an odd function solution $u^o(x)$ since the differential operator is invariant under changing of the variable $x\mapsto -x$. We can make stronger statement regarding the eigenvalue problem for even/odd functions.

Theorem 2. *The eigenvalue problem (1.1) for an odd function is equivalent to the problem*

$$H(w, D_w)f(w) = 0, \qquad f \in \mathcal{O}(\Omega),$$

while that for an even function is equivalent to the problem

$$P^e(w, D_w)f(w) = 0, \qquad f \in \mathcal{O}(\Omega).$$

Here $H(w, D_w)$ is Heun's differential operator (3.4), that is, a Fuchsian second-order differential operator with four regular singularities. The operator $P^e(w, D_w)$ is also a Fuchsian differential operator with four regular singularities (of third-order) and has an expression (4.1) in terms of Heun's operator.

In [PW], they construct the eigenfunctions and eigenvalues in terms of continued fractions determined by some three-term recurrence relation. Then this expression is a limit in some Hilbert space and is functional-analytic. Our expression given above is more complex-analytic, or even topological in the sense that the eigenvalues and eigenfunctions are determined by the monodromy representations of the Heun's operator.

Notice that the non-commutative harmonic oscillator introduced here gives a counterexample to the naive impression that an eigenfunction of the operator naturally arising in representation theory can be expressed in terms of hypergeometric functions. In fact, the Heun's operator for general parameters is far from 'hypergeometric'. For example, its solution is not of hypergeometric type, and the structure of the monodromy representation is different from that of hypergeometric equations.

This paper is organized as follows. In Section 2.1, we define coordinates in $\mathbf{C}^2$ appropriate for the investigation. This step is fairly easy, but the transformation in Lemma 4(ii) is a key to the analysis. In Section 2.2, we introduce several representations of the three dimensional simple Lie algebra $\mathfrak{sl}_2$. We explain the transformation T_6 in detail, though these seem standard in representation theory of semisimple Lie algebra, in order to give a formula for the inner product, which plays an important role in determining the eigenfunction. Section 2.3 represents the standard process of obtaining a single equation from a system of differential equations. In the final subsection, we transform the third-order operator into a second-order operator. In the context of classical analysis, this transformation can be expressed in terms of the (modified) Laplace transformation. In terms of the representations of $\mathfrak{sl}_2$, the third-order operator exists in the harmonic oscillator representation, while the second-order operator exists in the flat picture of the principal series representation. This fancy terminology is explained in Section 2.4.

In Section 3.1, we introduce Heun's operator and its relation to the differential operator under consideration. In Section 3.2, the inhomogeneous second-order differential equation (2.12) is proved to be equivalent to the third-order homogeneous equation (1.2). In Section 3.3, which is central in this paper, we translate the L^2-condition of an eigenfunction u into a holomorphicity condition of its Laplace transform $\hat{u}$. The idea here is fairly simple, and the task is to obtain a precise formulation. We next complete the proof of Theorem 1. In the last subsection, we calculate the index of the operator.

In Section 4, we study the ordinary differential equation (1.2). We can state this problem in several (equivalent) forms, but here we choose a formulation using Heun's operator. In Section 4.1, we summarize the properties of the operator $P^e(w, D_w)$, and prove Theorem 2 for even eigenfunctions. In Section 4.2, we prove the equivalence of the eigenvalue problem and the connection problem. In particular, we find that the spectrum is given by the zeros of the special connection coefficient $\eta_{ij}(\mu)$ of the ordinary differential equation $P^e(w, D_w)f(w) = 0$. In Section 4.3, we give a proof of Theorem 2 for odd eigenfunctions. In Section 4.4, we introduce a monodromy representation and prove that the eigenvalue problem is equivalent to the problem of determining the set of invariants of the restricted monodromy representation. This description of a spectrum depends only on the topological data, the monodromy, of the ordinary differential equations.

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2. Several equivalent forms of the problem

2.1. Parmeggiani and Wakayama's trick. We begin with the eigenvalue problem

$$\left[A(-\frac{\partial_x^2}{2}+\frac{x^2}{2})+B(x\partial_x+\frac{1}{2})-\nu I\right]u(x)=0.$$

We denote the standard generators of the simple Lie algebra $\mathfrak{sl}_2$ by

$$H=\begin{bmatrix}1&0\\0&-1\end{bmatrix}, X^+=\begin{bmatrix}0&1\\0&0\end{bmatrix}, X^-=\begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

These satisfy the commutation relations

$$[H,X^+]=2X^+,[H,X^-]=-2X^-,[X^+,X^-]=H.$$

Next we define the oscillator representation π of $\mathfrak{sl}_2$ by

$$\pi(H)=x\partial_x+1/2,\quad \pi(X^+)=x^2/2,\quad \pi(X^-)=-\partial_x^2/2. \tag{2.1}$$

We also denote the algebra homomorphism from the universal enveloping algebra $U(\mathfrak{sl}_2)$ to the ring $\mathbf{C}[x,\partial_x]$ of differential operators by the same character π. Using this representation, the problem (1.1) can be stated as

$$\left[A\pi(X^++X^-)+B\pi(H)-\mu I\right]u(x)=0. \tag{2.2}$$

Remark 3. *Let us define the matrix*

$$J=\begin{bmatrix}0&-1\\1&0\end{bmatrix}.$$

Since B is a skew symmetric matrix of order two, it is a multiple of J:

$$B=\mathrm{pf}(B)J,$$

where the Pfaffian $\mathrm{pf}(B)$ *is the lower-left entry of the skew symmetric matrix B. We note* $\mathrm{pf}(gB^tg)=(\det g)\,\mathrm{pf}(B)$.

Lemma 4.

(i) Diagonalization of A. *There is an orthogonal matrix $g_1 \in SO(2)$ such that $g_1 A g_1^{-1}$ is a diagonal matrix. Then we see that $g_1 B g_1^{-1} = B$.*

(ii) Parmeggiani and Wakayama's trick [PW, Corollary 4.2]. *Let $g_2 = (g_1 A g_1^{-1})^{1/2} = \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{bmatrix}$ for $g_1 A g_1^{-1} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$. Then*

$$g_2^{-1} g_1 A g_1^{-1} g_2^{-1} = I, \quad g_2^{-1} B g_2^{-1} = \frac{1}{\sqrt{\det A}} B, \quad g_2^{-1} I g_2^{-1} = (g_1 A g_1^{-1})^{-1}.$$

(iii) Cayley transformation. *Define a unitary matrix*

$$g_3 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \in U(2).$$

Then

$$g_3 J g_3^{-1} = -iH, \qquad g_3 g_1 A^{-1} g_1^{-1} g_3^{-1} = \delta \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}.$$

Here we define $\delta := \frac{1}{2} \operatorname{Tr}(A^{-1}) = \frac{\operatorname{Tr}(A)}{2 \det A}$, and $\varepsilon := -\frac{\operatorname{Tr}(g_1 A g_1^{-1} H)}{\operatorname{Tr}(A)}$.

Now, since $\operatorname{Tr}(g_1 A g_1^{-1} H)^2 = \operatorname{Tr}(A)^2 - 4 \det A$, we have the relations $\varepsilon^2 = 1 - \frac{4 \det A}{\operatorname{Tr}(A)^2}$, and $(1 - \varepsilon^2)\delta^2 \det A = 1$.

Corollary 5. *Let us define*

$$S_\pm := X^+ + X^- \pm i \frac{\operatorname{pf}(B)}{\sqrt{\det A}} H \in \mathfrak{sl}_2.$$

Then the problem (2.2) is equivalent to

$$\begin{bmatrix} \pi(S_-) - \mu\delta & \varepsilon\mu\delta \\ \varepsilon\mu\delta & \pi(S_+) - \mu\delta \end{bmatrix} u^{(3)}(x) = 0. \tag{2.3}$$

Here $u^{(3)} = g_3 g_2 g_1 u$.

As a corollary of this proposition, we prove the positivity of the eigenvalue, $\mu > 0$, in Section 4.5.

2.2. Inner automorphisms of $\mathfrak{sl}_2$. We change the representation π of the Lie algebra $\mathfrak{sl}_2$ into another representation appropriate for the purpose. Let us introduce the parameter κ defined by the relation $\operatorname{pf}(B)/\sqrt{\det A} = \tanh \kappa$. In other words, $\kappa = \frac{1}{2} \log\left(\frac{\sqrt{\det A} + \operatorname{pf}(B)}{\sqrt{\det A} - \operatorname{pf}(B)}\right)$. We consider

$$S_\pm = X^+ + X^- \pm i \frac{\operatorname{pf}(B)}{\det A} H = X^+ + X^- \pm i(\tanh \kappa) H \in \mathfrak{sl}_2. \tag{2.4}$$

Lemma 6. *We define*

$$g_7 = \begin{bmatrix} \frac{i-\sinh\kappa}{\cosh\kappa} & 0 \\ 0 & 1 \end{bmatrix}, g_6 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, g_5 = \begin{bmatrix} 1 & i\sinh\kappa \\ 0 & 1 \end{bmatrix}, g_4 = \begin{bmatrix} \cosh\kappa & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, for $g = g_7 g_6 g_5 g_4$, we have

$$gS_- g^{-1} = (\operatorname{sech}\kappa)H, \quad gS_+ g^{-1} = (\operatorname{sech}\kappa)\begin{bmatrix} \cosh 2\kappa & -\sinh 2\kappa \\ \sinh 2\kappa & -\cosh 2\kappa \end{bmatrix}.$$

Proof. This can be obtained easily by using the relations

$$igHg^{-1} = \begin{bmatrix} \sinh\kappa & -\cosh\kappa \\ \cosh\kappa & -\sinh\kappa \end{bmatrix}, \quad g(X^+ + X^-)g^{-1} = \begin{bmatrix} \cosh\kappa & -\sinh\kappa \\ \sinh\kappa & -\cosh\kappa \end{bmatrix}. \quad \square$$

We define the representations of $\mathfrak{sl}_2$ by

$$\pi_4(?) := \pi(g_4 ? g_4^{-1}),$$
$$\pi_5(?) := \pi(g_5 g_4 ? g_4^{-1} g_5^{-1}).$$

Then, we define the linear transformations $T_4, T_5 : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by

$$(T_4 f)(x) := (\cosh\kappa)^{1/4} f(\sqrt{\cosh\kappa}x),$$
$$(T_5 f)(x) := e^{i(\sinh\kappa)x^2/2} f(x).$$

The map T_4 corresponds to the matrix $g_4 \in GL(2,\mathbf{R})$, and represents the dilation of the variable. The map T_5 corresponds to the matrix $g_5 = \exp(i\sinh\kappa X_+) \in GL(2,\mathbf{C})$, and is equivalent to multiplication by the function $e^{i(\sinh\kappa)x^2/2}$. The maps T_4 and T_5 preserve the standard inner product on $L^2(\mathbf{R})$. We see that the map T_4 intertwines the action of $\mathfrak{sl}_2$ with π and π_4. Similarly, T_5 does that with π_4 and π_5. Graphically, we have the commutative diagram

$$\begin{array}{ccccc} L^2(\mathbf{R}) & \xrightarrow{T_4} & L^2(\mathbf{R}) & \xrightarrow{T_5} & L^2(\mathbf{R}) \\ \pi(?) \downarrow & & \pi_4(?) \downarrow & & \pi_5(?) \downarrow \\ L^2(\mathbf{R}) & \xrightarrow{T_4} & L^2(\mathbf{R}) & \xrightarrow{T_5} & L^2(\mathbf{R}). \end{array}$$

Next, we consider the transformation T_6 that corresponds to the matrix $g_6 \in SL(2,\mathbf{R})$. In the space $L^2(\mathbf{R})$, we have the inner product $(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$. We define the annihilation operator $\psi = (x + \partial_x)/\sqrt{2}$, and creation operator $\psi^+ = (x - \partial_x)/\sqrt{2}$. Then we have $[\psi, \psi^+] = 1$. The function $\varphi_0(x) := e^{-x^2/2}$ is the ground state, and by definition, $\psi\varphi_0 = 0$. In general, we define $\varphi_n = (\psi^+)^n \varphi_0$. Then the set $\{\varphi_n \mid n \in \mathbf{Z}_+\}$ constitutes an orthogonal basis with inner product $(\varphi_n, \varphi_n) = \sqrt{\pi}n!$. We denote the set of all finite linear combinations of the φ_n by $L^2(\mathbf{R}_x)_{\text{fin}}$. We also denote the set of all polynomials in one variable y by $\mathbf{C}[y]$.

The transformation T_6 mentioned above is a linear map

$$T_6 : L^2(\mathbf{R})_{\mathrm{fin}} \to \mathbf{C}[y]$$

and we defined it by $T_6(\varphi_n) = y^n$. We see that $T_6(\psi^+\varphi) = yT_6(\varphi)$ and $T_6(\psi\varphi) = \partial_y T_6(\varphi)$.

Recall the definition (2.1) of the representation π of the Lie algebra $\mathfrak{sl}_2$ on $L^2(\mathbf{R})_{\mathrm{fin}}$

$$\pi(H) = x\partial_x + \frac{1}{2}, \pi(X^+) = x^2/2, \pi(X^-) = -\partial_x^2/2.$$

We define the representation of $\mathfrak{sl}_2$ on $\mathbf{C}[y]$ by

$$\pi_6(H) = y\partial_y + \frac{1}{2}, \pi_6(X^+) = y^2/2, \pi_6(X^-) = -\partial_y^2/2. \tag{2.5}$$

Then, we calculate

$$\begin{cases} \pi(g_6^{-1}Hg_6) = \pi(X^+ + X^-) = (x^2 - \partial_x^2)/2 = \psi^+\psi + \frac{1}{2}, \\ \pi(g_6^{-1}X^+g_6) = \pi(\frac{1}{2}(-H + X^+ - X^-)) = (\psi^+)^2/2, \\ \pi(g_6^{-1}X^-g_6) = \pi(\frac{1}{2}(-H - X^+ + X^-)) = -\psi^2/2. \end{cases}$$

This proves that

$$\pi'(g_6?g_6^{-1})T_6 = T_6\pi(?),$$

which implies that

$$\begin{array}{ccc} L^2(\mathbf{R})_{\mathrm{fin}} & \xrightarrow{T_6} & \mathbf{C}[y] \\ \pi(?)\downarrow & & \downarrow \pi'(g_6?g_6^{-1}) \\ L^2(\mathbf{R})_{\mathrm{fin}} & \xrightarrow{T_6} & \mathbf{C}[y]. \end{array}$$

Next, we introduce an inner product on $\mathbf{C}[y]$ such that the map T_6 is an isometry with respect to this inner product. To be more explicit, the inner product should be $(y^m, y^n) = \delta_{mn}\sqrt{\pi}n!$. This can be realized as

$$(f, g) = (f(\partial_y)\bar{g}(y))|_{y=0}, \qquad f, g \in \mathbf{C}[y].$$

If we denote the completion of $\mathbf{C}[y]$ with respect to this inner product by $\overline{\mathbf{C}[y]}$, then the map T_6 can be extended to the isometry between the Hilbert spaces $L^2(\mathbf{R})$ and $\overline{\mathbf{C}[y]}$.

Finally, we define the representation of $\mathfrak{sl}_2$ on $\mathbf{C}[y]$ by

$$\begin{aligned} \pi_6(?) &:= \pi'(g_6g_5g_4?g_4^{-1}g_5^{-1}g_6^{-1}), \\ \pi_7(?) &:= \pi'(g?g^{-1}), \qquad \text{recall } g = g_7g_6g_5g_4, \end{aligned}$$

and the intertwiner $T_7 : \mathbf{C}[y] \to \mathbf{C}[y]$ by

$$(T_7f)(y) := f(\sqrt{\tfrac{i - \sinh\kappa}{\cosh\kappa}}y).$$

The map T_7 corresponds to the matrix $g_7 \in GL(2, \mathbf{C})$ and preserves the inner product on $\mathbf{C}[y]$. We thus obtain the commutative diagram

$$\begin{array}{ccccc} L^2(\mathbf{R})_{\mathrm{fin}} & \xrightarrow{T_6} & \mathbf{C}[y] & \xrightarrow{T_7} & \mathbf{C}[y] \\ \pi_5(?)\downarrow & & \pi_6(?)\downarrow & & \pi_7(?)\downarrow \\ L^2(\mathbf{R})_{\mathrm{fin}} & \xrightarrow{T_6} & \mathbf{C}[y] & \xrightarrow{T_7} & \mathbf{C}[y]. \end{array}$$

Corollary 7. *With the notation above, we have the expression*

$$\left\{\begin{aligned} \pi_7(S_-) &= (\operatorname{sech}\kappa)\pi'(H), \\ \pi_7(S_+) &= (\operatorname{sech}\kappa)\pi'((\cosh 2\kappa)H - (\sinh 2\kappa)(X^+ - X^-)). \end{aligned}\right. \tag{2.6}$$

The equation (2.3) is equivalent to

$$\begin{bmatrix} \pi_7(S_-) - \mu\delta & \varepsilon\mu\delta \\ \varepsilon\mu\delta & \pi_7(S_+) - \mu\delta \end{bmatrix} u^{(7)}(x) = 0, \tag{2.7}$$

where $u^{(7)} = T_7T_6T_5T_4u^{(3)}$.

2.3. Reduction of a system of equations to a single equation. This is a standard argument.

Lemma 8. *Suppose* $\varepsilon\mu \neq 0$. *Then the system of differential equations (2.7),*

$$\begin{bmatrix} \pi_7(S_-) - \mu\delta & \varepsilon\mu\delta \\ \varepsilon\mu\delta & \pi_7(S_+) - \mu\delta \end{bmatrix} \begin{bmatrix} u_-^{(7)}(x) \\ u_+^{(7)}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

is equivalent to the single differential equation

$$\left[(\pi_7(S_+) - \mu\delta)(\pi_7(S_-) - \mu\delta) - (\varepsilon\mu\delta)^2\right] u_-^{(7)}(x) = 0. \tag{2.8}$$

Proof. $u_+ = -(\pi_7(S_-) - \mu\delta)(u_-)/(\varepsilon\mu\delta)$. $\square$

Corollary 9. *Let us define an element R of the universal enveloping algebra as*

$$R := (2(X^+ - X^-) - 2(\coth 2\kappa)H + \tfrac{\mu\delta}{\sinh\kappa})(H - \mu\delta\cosh\kappa) + (\varepsilon\mu\delta)^2\coth\kappa \in U(\mathfrak{sl}_2).$$

Then, $(\pi_7(S^+) - \mu\delta)(\pi_7(S_-) - \mu\delta)) - (\varepsilon\mu\delta)^2 = -(\tanh\kappa)\pi'(R)$.

We define $\nu := \mu\delta\cosh\kappa$, for simplicity. Then the operator R can be written as

$$R = (2(X^+ - X^-) - 2(\coth 2\kappa)H + \tfrac{2\nu}{\sinh 2\kappa})(H - \nu) + \tfrac{2(\varepsilon\nu)^2}{\sinh 2\kappa} \in U(\mathfrak{sl}_2). \tag{2.9}$$

For convenience, we next give an expression for the coefficients of R in terms of the matrices A and B and the eigenvalue μ:

$$\tfrac{2(1-\varepsilon^2)}{\sinh 2\kappa} = \tfrac{4(\det A - \det B)\sqrt{\det A}}{\operatorname{Tr}(A)^2\operatorname{pf}(B)}, \quad 2\coth 2\kappa = \tfrac{\det A + \det B}{\sqrt{\det A}\operatorname{pf}(B)}, \quad \nu = \tfrac{\operatorname{Tr}(A)}{2\sqrt{(\det A - \det B)\det A}}\mu.$$

2.4. Laplace transform. We recall the realization given in (2.5):

$$\left\{\begin{aligned} \pi'(H) &= y\partial_y + 1/2, \\ \pi'(X^+) &= y^2/2, \\ \pi'(X^-) &= -\partial_y^2/2. \end{aligned}\right.$$

We now introduce a second realization (representation) of the $\mathfrak{sl}_2$-triple:

$$\left\{\begin{aligned} \varpi(H) &= z\partial_z + 1/2 = \theta_z + 1/2, \\ \varpi(X^+) &= z^2(\tfrac{1}{2}z\partial_z + 1) = z^2(\tfrac{1}{2}\theta_z + 1), \\ \varpi(X^-) &= -\tfrac{1}{2z}\partial_z + \tfrac{1}{2z^2} = -\tfrac{1}{2z^2}(\theta_z - 1). \end{aligned}\right. \tag{2.10}$$

Here, $\theta_z = z\partial_z$ denotes Euler's degree operator. The relation between these two realizations is given by the following.

Proposition 10. *We define the (modified) Laplace transform*

$$\hat{u}(z) := \int_0^\infty u(yz)e^{-y^2/2}ydy = z^{-2}\int_0^{z\infty} u(y)e^{-y^2/(2z^2)}ydy.$$

This is a linear map from y-space to z-space with the following properties:

(i) *It gives $y^n \mapsto \Gamma(\frac{n}{2}+1)(\sqrt{2}z)^n$ for any $n \in \mathbf{Z}_+$. In particular, $1 \mapsto 1$.*

(ii) *Suppose the expansion $u = \sum_{n=0}^\infty u_n y^n \in \mathbf{C}[y]$, then $\hat{u}(z) = \sum_{n=0}^\infty u_n \Gamma(\frac{n}{2}+1)(\sqrt{2}z)^n$.*

(iii) *The Laplace transform almost intertwines the action of $\mathfrak{sl}_2$:*

$$\left\{\begin{aligned} (\pi'(H)u)\hat{} &= \varpi(H)(\hat{u}) \\ (\pi'(X^+)u)\hat{} &= \varpi(X^+)(\hat{u}) \\ (\pi'(X^-)u)\hat{} &= \varpi(X^-)(\hat{u}) - \tfrac{1}{2}u(0)z^{-2}. \end{aligned}\right. \tag{2.11}$$

(iv) *If we define the inner product in z-space such that $\{z^n \mid n \in \mathbf{Z}_+\}$ forms an orthogonal basis and $(z^n, z^n) = \Gamma(\frac{n+1}{2})/\Gamma(\frac{n}{2}+1)$, then the Laplace transformation is an isometry.*

Proof. (iii) This is proved by the calculation

$$\begin{aligned} (\pi'(X^-)u)\hat{}(z) &= \int_0^\infty -\frac{1}{2}u''(yz)e^{-y^2/2}ydy = \frac{1}{2z}\int_0^\infty u'(yz)(e^{-y^2/2}y)'dy, \\ -\frac{1}{2z}\partial_z\hat{u}(z) &= -\frac{1}{2z}\int_0^\infty u'(yz)e^{-y^2/2}y^2dy, \\ \frac{1}{2z^2}\hat{u}(z) &= -\frac{1}{2z^2}\int_0^\infty u(yz)(e^{-y^2/2})'dy \\ &= -\frac{1}{2z^2}\left([u(yz)e^{-y^2/2}]_{y=0}^{y=\infty} - \int_0^\infty zu'(yz)e^{-y^2/2}dy\right). \end{aligned}$$

(iv) Since $(y^n, y^n) = \sqrt{\pi}n!$, we have

$$(z^n, z^n) = \frac{\sqrt{\pi}n!}{2^n\Gamma(\frac{n}{2}+1)^2} = \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{2^n\Gamma(\frac{n}{2}+1)^2} = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)}. \qquad \square$$

Note that $\Gamma(\frac{n+1}{2})/\Gamma(\frac{n}{2}+1) = \frac{2}{\sqrt{\pi}}\int_0^{\pi/2}\sin^n x dx$. Since $\Gamma(z+a)/\Gamma(z+b) \sim z^{a-b}$, asymptotically for large n, we have $(z^n, z^n) \sim \sqrt{2/n}$.

The property (iii) means that the representations π' and ϖ do not intertwine, but almost. The precise relation is given as follows. In fact, the Casimir operator $4X^+X^- + H^2 - 2H \in U(\mathfrak{sl}_2)$ takes the value $-3/4$ in both representations. Let us denote the set of all Laurent polynomials (finite linear combinations of $\{z^n \mid n \in \mathbf{Z}\}$) by $\mathbf{C}[z, z^{-1}]$. The subspace $\mathbf{C}[z^2, z^{-2}]$ consists of the set of all even functions in $\mathbf{C}[z, z^{-1}]$. The set of even polynomials is written $\mathbf{C}[z^2]$, while the set of odd polynomials is written $z\mathbf{C}[z^2]$.

By the definition of ϖ, (2.10), we see that the representation $(\varpi, \mathbf{C}[z, z^{-1}])$ has a subrepresentation $(\varpi, z^{-2}\mathbf{C}[z^{-2}])$. The representation $(\pi', \mathbf{C}[y^2])$ is isomorphic to the quotient representation $(\varpi, \mathbf{C}[z^2, z^{-2}]/z^{-2}\mathbf{C}[z^{-2}])$. In other words, the even part $(\pi', \mathbf{C}[y^2])$ is the Langlands quotient of the representation $(\varpi, \mathbf{C}[z^2, z^{-2}])$. On the other hand, $(\varpi, z\mathbf{C}[z^2])$ is a subrepresentation of $(\varpi, \mathbf{C}[z^2, z^{-2}])$. This subrepresentation is isomorphic to the odd part $(\pi', y\mathbf{C}[y^2])$.

Corollary 11. *The operator $\pi'(R)$ satisfies the relation*

$$(\pi'(R)u)^{\widehat{}} = \varpi(R)(\hat{u}) + (\frac{1}{2} - \nu)\hat{u}(0)z^{-2}.$$

It follows from this corollary that equation (2.8) is equivalent to

$$\varpi(R)(\hat{u}) + (\frac{1}{2} - \nu)\hat{u}(0)z^{-2} = 0. \tag{2.12}$$

Remark. Parmeggiani and Wakayama [PW] have already obtained a part of the results in this section, and stated in the different terminology. For example, their recurrence equations (13) in [PW] corresponds to the differential equation (2.12). However, the Laplace transformation seems new.

3. Laplace transform of an eigenfunction

3.1. Heun's differential operator.

Substituting the realization (2.10) into (2.9), we obtain the realization

(3.1)
$$\varpi(R) = \left(z^2(\theta_z + 2) + z^{-2}(\theta_z - 1) - 2(\coth 2\kappa)(\theta_z + \tfrac{1}{2}) + \tfrac{2\nu}{\sinh 2\kappa}\right)(\theta_z + \tfrac{1}{2} - \nu) + \tfrac{2(\varepsilon\nu)^2}{\sinh 2\kappa}$$

(3.2)
$$\varpi(R) = \left((z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \tfrac{1}{2}) + \tfrac{3}{2}(z^2 - z^{-2}) + \tfrac{2\nu}{\sinh 2\kappa}\right)(\theta_z + \tfrac{1}{2} - \nu) + \tfrac{(\varepsilon\nu)^2}{\sinh 2\kappa}.$$

Conjugating by z, we have

$$z^{-1}\varpi(R)z = \left((z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \tfrac{3}{2}) + \tfrac{3}{2}(z^2 - z^{-2}) + \tfrac{2\nu}{\sinh 2\kappa}\right)(\theta_z + \tfrac{3}{2} - \nu) + \tfrac{(\varepsilon\nu)^2}{\sinh 2\kappa}.$$

Since the operators $\varpi(H)$, $\varpi(X^+)$ and $\varpi(X^-)$ are invariant under the symmetry $z \mapsto -z$, the operator $\varpi(R)$ is invariant under the same symmetry. This can be seen in expression (3.1) or (3.2). This implies that the operator $\varpi(R)$ can be written in terms of the variable z^2. Now let us introduce the new variable $w := z^2 \coth\kappa$. Then, factoring by the leading coefficient, from the above expression, we obtain

$$z^{-1}\varpi(R)z = 4(\tanh\kappa)w(w-1)(w - \coth^2\kappa)H(w, D_w), \tag{3.3}$$

where, $H(w, D_w)$ is Heun's differential operator [H, (1.1.1)],

$$H(w, D_w) := \partial_w^2 + (\frac{\gamma'}{w} + \frac{\delta'}{w-1} + \frac{\epsilon'}{w-a'})\partial_w + \frac{\alpha'\beta' w - q'}{w(w-1)(w-a')}, \tag{3.4}$$

with the parameters

$$\gamma' = \frac{7-2\nu}{4}, \delta' = \frac{3-2\nu}{4}, \epsilon' = \frac{3+2\nu}{4}, \alpha' = \frac{3}{2}, \beta' = \frac{3-2\nu}{4}, a' = \coth^2\kappa$$

and

$$q' = \frac{4\nu^2(1-\varepsilon^2) - 12\nu\cosh^2\kappa + 9\cosh 2\kappa}{16\sinh^2\kappa}.$$

Recall that

$$1-\varepsilon^2=\frac{4\det A}{\operatorname{Tr}(A)^2},\quad \coth^2\kappa=\frac{\det A}{\det B},\quad \nu=\frac{\operatorname{Tr}(A)}{2\sqrt{(\det A-\det B)\det A}}\mu.$$

In particular, $H(w,D_w)$ is a second-order linear differential operator with four regular singular points on the Riemannian sphere. In terms of the Riemann schema (a P-symbol), the list of the exponents [H, (1.1.3)] is given as

$$\begin{pmatrix} 0 & 1 & a' & \infty & & \\ 0 & 0 & 0 & \alpha' & w & q' \\ 1-\gamma' & 1-\delta' & 1-\epsilon' & \beta' & & \end{pmatrix}=\begin{pmatrix} 0 & 1 & \coth^2\kappa & \infty & & \\ 0 & 0 & 0 & \frac{3}{2} & w & q' \\ \frac{2\nu-3}{4} & \frac{2\nu+1}{4} & \frac{-2\nu+1}{4} & \frac{-2\nu+3}{4} & & \end{pmatrix}.$$

In particular, the parameter κ designates the locations of the singular points, the parameter ν designates the exponents, and the parameter ε is an accessory parameter.

The above discussion gives the following:

Proposition 12. *The operator $\varpi(R)$ has the following properties:*

(i) *It is a second-order linear ordinary differential operator and has rational coefficients in z.*
(ii) *It has six singular points $z=0,\pm\sqrt{\tanh\kappa},\pm\sqrt{\coth\kappa},\infty$.*
(iii) *All singularities are regular singularities.*
(iv) *The exponents at $z=0$ are 1 and $(2\nu-1)/2$. Those at $z=\infty$ are 2 and $-(2\nu-1)/2$. Those at the points $z=\pm\sqrt{\tanh\kappa}$ are 0 and $(2\nu+1)/4$. Those at the points $z=\pm\sqrt{\coth\kappa}$ are 0 and $(-2\nu+1)/4$.*

Proof. By the conjugation of $w^{1/2}$, the operator $\varpi(R)$ has the P-symbol

$$\begin{pmatrix} 0 & \tanh\kappa & \coth\kappa & \infty & \\ \frac{1}{2} & 0 & 0 & 1 & z^2 \\ \frac{2\nu-1}{4} & \frac{2\nu+1}{4} & \frac{-2\nu+1}{4} & \frac{-2\nu+1}{4} & \end{pmatrix}.$$

By unfolding $z^2\mapsto z$, this P-symbol is equal to

$$\begin{pmatrix} 0 & \sqrt{\tanh\kappa} & -\sqrt{\tanh\kappa} & \sqrt{\coth\kappa} & -\sqrt{\coth\kappa} & \infty & \\ 1 & 0 & 0 & 0 & 0 & 2 & z \\ \frac{2\nu-1}{2} & \frac{2\nu+1}{4} & \frac{2\nu+1}{4} & \frac{-2\nu+1}{4} & \frac{-2\nu+1}{4} & \frac{-2\nu+1}{2} & \end{pmatrix}.$$

This proves the assertion (iv). □

3.2. The differential equation.

We define the third-order differential operator $P(z,D_z):=(z\partial_z+2)\varpi(R)=\varpi((H+\frac{3}{2})R)$. This is the operator in Theorem 1 in the Introduction.

Lemma 13. *The following conditions for a holomorphic function $\hat{u}(z)\in\mathcal{O}_0$ at the origin are equivalent:*

(i) $\varpi(R)(\hat{u})+(\frac{1}{2}-\nu)\hat{u}(0)z^{-2}=0$ *(that is, the condition (2.12)).*
(ii) $(z^2\varpi(R))\hat{u}+(\frac{1}{2}-\nu)\hat{u}(0)=0$.
(iii) $\partial_z z^2\varpi(R)\hat{u}=0$.
(iv) $P(z,D_z)\hat{u}=0$.

Proof. The relations (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) are clear. Since the constant term on the left-hand side of (ii) vanishes for an arbitrary holomorphic function $\hat{u}(z) \in \mathcal{O}_0$, the condition (iii) implies (ii). $\square$

Now we summarize the properties of the operator $P(z, D_z)$. We see that the operator $P(z, D_z)$ is holomorphic on $\mathbf{C}$. This follows from (3.1) and the formula $(\theta_z+2)z^{-2}(\theta_z - 1) = \partial_z^2$. Also, the coefficient of ∂_z^3 of $P(z, D_z)$ is $z^3(z^2 + z^{-2} - 2\coth 2\kappa) = z(z^2 - \tanh\kappa)(z^2 - \coth\kappa)$. The order of the zeros of this coefficient at the singular points $z = 0$, $\pm\sqrt{\tanh\kappa}$ and $\pm\sqrt{\coth\kappa}$ is 1.

3.3. L^2-conditions and analytic continuation. We now consider the holomorphic solution $\hat{u}(z) \in \mathcal{O}_0$ satisfying the equation $P(z, D_z)\hat{u} = 0$ (see Lemma 13). The solution can be analytically continued along a path avoiding the singular points of the differential equation. In particular, the solution is holomorphic on the (open) disk of radius $\sqrt{|\tanh\kappa|}$. We consider the behavior of the solution near the points $z = \pm\sqrt{\tanh\kappa}$. There are two possibilities:

- If a solution $\hat{u}(z)$ is holomorphic near the points $z = \pm\sqrt{\tanh\kappa}$, then it is continued to a (single-valued) holomorphic function on the disk $\{z \in \mathbf{C} \mid |z| < \sqrt{|\coth\kappa|}\}$ centered at the origin. In this case, the Taylor expansion $\hat{u}(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies the following asymptotic property:

$$\forall\varepsilon_1 > 0, \exists N \text{ such that } \forall n > N, \qquad |a_n| \le (|\tanh\kappa| + \varepsilon_1)^{n/2}.$$

- If a solution $\hat{u}(z)$ cannot be holomorphically continued to at least one point $z = \pm\sqrt{\tanh\kappa}$, then the radius of convergence of the Taylor series $\hat{u}(z) = \sum_{n=0}^{\infty} a_n z^n$ is $\sqrt{|\tanh\kappa|}$. In this case,

$$\forall\varepsilon_1 > 0 \text{ and } \forall N, \exists n > N \text{ such that } \qquad |a_n| \ge (|\coth\kappa| - \varepsilon_1)^{n/2}.$$

Proposition 14. *Consider a formal power series solution $u(y) = \sum_{n=0}^{\infty} u_n y^2 \in \mathbf{C}[[y]]$ of the equation $\pi'(R)u = 0$. Its Laplace transform is written as $\hat{u}(z) \in \mathbf{C}[[z]]$. Then, we have the following:*

(i) *Since the operator is regular singular at the origin, any formal power series solution $\hat{u}(z) \in \mathbf{C}[[z]]$ of the equation $P(z, D_z)\hat{u} = 0$ converges to a holomorphic function near the origin.*

(ii) *The following conditions are equivalent:*
  (a) *$u(y)$ converges in the Hilbert space $\overline{\mathbf{C}[y]}$.*
  (b) *$\hat{u}(z)$ converges to a holomorphic function on the unit disk.*
  (c) *Both the even part $\hat{u}^e(z)$ and the odd part $\hat{u}^o(z)$ of $\hat{u}(z)$ can be holomorphically continued to a neighbourhood of the closed interval $[0, \sqrt{\tanh\kappa}]$.*

(iii) *The following conditions are equivalent:*
  (a) *$u(y)$ does not converge in the Hilbert space $\overline{\mathbf{C}[y]}$.*

(b) $\hat{u}(z)$ *cannot be continued to a holomorphic function on the unit disk.*

(c) *Either the even part* $\hat{u}^e(z)$ *or the odd part* $\hat{u}^o(z)$ *of* $\hat{u}(z)$ *cannot be holomorphically continued to a neighbourhood of the closed interval* $[0, \sqrt{\tanh\kappa}]$.

Proof. We set $a_n = u_n\Gamma(\frac{n}{2}+1)2^{n/2}$. Then the condition (ii-b) and (iii-b) imply

$$|u_n|^2(y^n, y^n) = \frac{|a_n|^2\sqrt{\pi}n!}{\Gamma(\frac{n}{2}+1)^2 2^n}\begin{cases} \leq (|\tanh\kappa| + \varepsilon_1)^n \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} & \text{for case (ii)}, \\ \geq (|\coth\kappa| - \varepsilon_1)^n \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} & \text{for case (iii)}. \end{cases} \tag{3.5}$$

This proves the equivalences (ii-a) $\Leftrightarrow$ (ii-b) and (iii-a) $\Leftrightarrow$ (iii-b). Then, (ii-b) implies (ii-c) by the relations $\hat{u}^e(z) = (\hat{u}(z) + \hat{u}(-z))/2$, $\hat{u}^o(z) = (\hat{u}(z) - \hat{u}(-z))/2$. Conversely, the condition (iii-c) implies that the even function $\hat{u}^e(z)$ is also holomorphic at $z = -\sqrt{\tanh\kappa}$. Since this function is a solution of the differential equation, it can be holomorphically continued to regular points. Thus, it is holomorphic on the unit disk. The same holds for the odd part $\hat{u}^o(z)$. This proves (ii-c) for $\hat{u} = \hat{u}^e + \hat{u}^o$. □

Remark 15. *If we replace the Hilbert space* $L^2(\mathbf{R})$ *by some Sobolev-type space, then the estimate (3.5) requires some additional factors such as* $(1+n^2)^\sigma$. *This change affects neither the statement nor the proof of the proposition.*

Remark 16. *The unit disk can be replaced by a connected and simply-connected open subset* Ω' *of* $\mathbf{C}$ *which contains the three points* $0, \pm\sqrt{\tanh\kappa}$ *and which contains neither of the points* $\pm\sqrt{\coth\kappa}$. *In fact, Proposition 14 remains valid if we replace the unit disk by such a domain in the statements of conditions (ii-b) and (iii-b).*

We denote the set of holomoprhic functions on Ω' by $\mathcal{O}(\Omega')$. The germ of holomorphic functions at the origin is denoted by $\mathcal{O}_0$.

To summarize the above discussion, the spectral problem (1.1) is equivalent to that of finding all the holomorphic solutions $U(z) \in \mathcal{O}(\Omega')$ of the differential equation $P(z, D_z)U(z) = 0$.

3.4. Index of the operator.

Lemma 17. *The index of the map*

$$P(z, D_z) : \mathcal{O}(\Omega') \to \mathcal{O}(\Omega') \tag{3.6}$$

is zero.

Proof. The index is, by definition, the difference of the dimensions of the kernel and cokernel. It is known (e.g., see [K]) that the map (3.6) is continuous, it has a closed range, the dimensions of its kernel and cokernel are both finite, and its index is equal to the difference of the order of the operator $P(z, D_z)$ and the number of zeros with multiplicity in the domain Ω'. Thus the index of the map in question is $3 - 1 \times 3 = 0$. □

Corollary 18. *The following conditions are equivalent:*

(i) *The map (3.6) is injective.*

(ii) *The map (3.6) is surjective.*

(iii) *The map (3.6) is bijective.*

If one takes a monomial basis $\{z^n \mid n \in \mathbf{Z}_+\}$ of $\mathbf{C}[[z]]$, the condition (iii) of Corollary 18 can be expressed in terms of the 'determinant' of a tridiagonal, by (3.1), matrix of infinite size. This determinant should coincide with the function f introduced in [PW], where this function is constructed with the limit of some continued fractions. If the determinant is zero, then we have an infinite series solution in $\mathbf{C}[[z]] = \hat{\mathcal{O}}_0$. Since the operator $P(z, D_z)$ has regular singularity at the origin (Proposition 14(i)), such a formal solution converges to a holomorphic solution at the origin.

4. Connection problem

In this section, we discuss the relation between our problem and the connection problem of a linear ordinary differential equation in a complex domain. It is instructive to relate our work with Heun's operator, and therefore we work in the variable w in this section. We set $\Omega' = \{z \in \mathbf{C} \mid |z| < 1\}$, $\Omega = \{w \in \mathbf{C} \mid |w| < |\coth \kappa|\}$.

4.1. Expression. Every solution $f \in \mathcal{O}(\Omega')$ of $P(z, D_z)f = 0$ is the sum of an even solution $f^e(z)$ and an odd solution $f^o(z)$. By the map $w = z^2 \coth \kappa$, the set of even functions in $\mathcal{O}(\Omega')$ is isomorphic to $\mathcal{O}(\Omega)$. The set of odd functions in $\mathcal{O}(\Omega')$ is identified with $\sqrt{w}\mathcal{O}(\Omega)$. We denote the differential operator $P(z, D_z)$ written in terms of w by $P^e(w, D_w)$. This operator acts on $\mathcal{O}(\Omega)$ and $\sqrt{w}\mathcal{O}(\Omega)$ as

$$P^e(w, D_w) : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega), \qquad P^e(w, D_w) : \sqrt{w}\mathcal{O}(\Omega) \to \sqrt{w}\mathcal{O}(\Omega).$$

Using the variable w, we have

$$\varpi(R) = \big(2w(\tanh\kappa)(\theta_w + 1) + w^{-1}(\coth\kappa)(2\theta_w - 1) - 2(\coth 2\kappa)(2\theta_w + \tfrac{1}{2}) + \tfrac{2\nu}{\sinh 2\kappa}\big) \times$$
$$(2\theta_w + \tfrac{1}{2} - \nu) + \tfrac{2(\varepsilon\nu)^2}{\sinh 2\kappa},$$

where $\theta_w = w\partial_w = \frac{1}{2}\theta_z$. By this expression $w\varpi(R)$ is holomorphic on $w \in \mathbf{C}$. We see that $P^e(w, D_w) = (\theta_z + 2)\varpi(R) = 2(\theta_w + 1)\varpi(R) = 2\partial_w w\varpi(R)$. In other words, $P^e(w, D_w)$ can be factorized into a product of the two differential operators ∂_w and $w\varpi(R)$ holomorphic on $\mathbf{C}$ in the variable w. By (3.3), we have

$$P^e(w, D_w) = 8(\tanh\kappa)\partial_w \cdot w^2(w-1)(w - \coth^2\kappa) \cdot \sqrt{w}H(w, D_w)\sqrt{w}^{-1}. \tag{4.1}$$

Thus the exponents of $P^e(w, D_w)$ are given by the P-symbol

$$\left\{\begin{array}{ccccc} 0 & 1 & \coth^2\kappa & \infty & \\ 0 & 1 & 1 & 2 & \\ \frac{1}{2} & 0 & 0 & 1 & w \\ \frac{\nu}{2} - \frac{1}{4} & \frac{\nu}{2} + \frac{1}{4} & -\frac{\nu}{2} + \frac{1}{4} & -\frac{\nu}{2} + \frac{1}{4} & \end{array}\right\}.$$

The new exponent is $2 - 2 = 0$ at $w = 0$, it is $2 - 1 = 1$ at $w = 1, \coth^2\kappa$, and it is $-(2-4) = 2$ at $w = \infty$.

4.2. Connection problem. We illustrate the corresponding connection problem briefly. For simplicity, we assume in this section that the parameter ν satisfies $\nu + \frac{1}{2} \notin \mathbf{Z}_+$.

First, consider the differential equation $w\varpi(R)U(w) = 0$. Choose a basis $\{f_{01}, f_{02}\}$ of solutions near $w = 0$ such that the exponent of f_{01} is $1/2$ and that of f_{02} is $(2\nu - 1)/4$. Then $\sqrt{w}^{-1} f_{01}$ is holomorphic and f_{02} is not. If $(2\nu - 1)/4 \notin \frac{1}{2} - \mathbf{Z}_+$, then f_{02} is ambiguous, but this is irrelevant to the present discussion.

Now, choose a basis $\{f_{11}, f_{12}\}$ of solutions near $w = 1$ such that the expoent of f_{11} is 0 and that of f_{12} is $(2\nu + 1)/4$. Then f_{11} is holomorphic and f_{12} is not.

Next we select a basis of the solution of the differential equation $P^e(w, D_w)U(w) = 0$. Let f_{00} be a solution with exponent 0 near $w = 0$. This solution is holomorphic. Then, let f_{10} be a solution with exponent 1 near $w = 1$. This is also holomorphic.

The space $\ker(P^e, \mathcal{O}_0)$ of holomorphic solutions at the origin is spanned by f_{00}. Similarly, the space $\ker(P^e, \sqrt{w}\mathcal{O}_0)$ is spanned by f_{01}. For the point $w = 1$, the space $\ker(P^e, \mathcal{O}_1)$ is spanned by f_{10} and f_{11}.

Now we consider the connection matrix along the open interval $(0, 1)$:

$$f_{0j} = \sum_{i=0}^{2} f_{1i}\eta_{ij}, \quad j = 0, 1, 2.$$

By definition, the function $\eta_{ij} = \eta_{ij}(\nu, \kappa, \varepsilon)$ does not depend on w, but it may depend on the parameters ν, κ, and ε. Since $\{f_{kj} \mid j = 1, 2\}$ forms a basis of solutions of the equation $w\varpi(R)U(w) = 0$, we have $\eta_{01} = \eta_{02} = 0$.

The restriction map from Ω to the origin gives a natural map $\ker(P^e, \mathcal{O}(\Omega)) \to \ker(P^e, \mathcal{O}_0)$. We characterize the image of this map by the connection coefficient.

Proposition 19.

(i) *The (even) function* $f_{00} \in \ker(P^e, \mathcal{O}_0)$ *comes from a function in* $\ker(P^e, \mathcal{O}(\Omega))$ *if and only if* $\eta_{20} = 0$.

(ii) *The (odd) function* $f_{01} \in \ker(P^e, \sqrt{w}\mathcal{O}_0)$ *comes from a function in* $\ker(P^e, \sqrt{w}\mathcal{O}(\Omega))$ *if and only if* $\eta_{21} = 0$.

In other words, provided that ν is "non-integral", then ν is an eigenvalue of the eigenvalue problem if and only if $\eta_{20}\eta_{21} = 0$. An expression for the connection coefficients is given in [SS1] and [SS2].

4.3. Odd case. For the odd case, the connection to Heun's operator is more direct. Let us consider

$$P^e(z, D_z) : \sqrt{w}\mathcal{O}(\Omega) \to \sqrt{w}\mathcal{O}(\Omega).$$

Lemma 20. *We have the isomorphism*

$$\ker(P^e, \sqrt{w}\mathcal{O}(\Omega)) = \sqrt{w}\ker(H(w, D_w), \mathcal{O}(\Omega)).$$

Proof. First, we see that $\sqrt{w}f \in \ker(P^e, \sqrt{w}\mathcal{O}(\Omega))$ if and only if

$$f \in \ker(\sqrt{w}^{-1} P^e(w, D_w)\sqrt{w}, \mathcal{O}(\Omega)).$$

Then, by (4.1), we have

$$\sqrt{w}^{-1} P^e(w, D_w)\sqrt{w} = 8(\tanh\kappa)\sqrt{w}^{-1}\partial_w\sqrt{w} \cdot w^2(w-1)(w - \coth^2\kappa)H(w, D_w).$$

Finally, since $\sqrt{w}^{-1} \cdot \partial_w \cdot \sqrt{w}$ is bijective on $\mathcal{O}(\Omega)$, the kernel of the operator is given by the kernel of $H(w, D_w)$. $\square$

Remark 21. *The operator*

$$w(1-w)H(w, D_w) : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega).$$

has index zero.

The exponents of this operator are, as is in Section 3.1, given by

$$\left\{ \begin{matrix} 0 & 1 & \coth^2 \kappa & \infty & \\ 0 & 0 & 0 & \frac{3}{2} & w \\ \frac{\nu}{2} - \frac{3}{4} & \frac{\nu}{2} + \frac{1}{4} & -\frac{\nu}{2} + \frac{1}{4} & -\frac{\nu}{2} + \frac{3}{4} & \end{matrix} \right\}.$$

Then if $(2\nu+1)/4 \notin \mathbf{Z}_+$, the eigenvalue problem remains equivalent to $\eta_{21} = 0$. In particular, this is the case for $(2\nu-1)/4 \in \mathbf{Z}_+$, which was excluded in Section 4.2.

4.4. Monodromy representation. In this subsection, we assume that $-(\nu + \frac{3}{2}) \notin \mathbf{Z}_+$. Since all the eigenvalues ν satisfy $\nu > 0$, as seen in Section 4.A, this assumption is harmless. We choose a base point w_0 such that $0 < w_0 < 1$ and fix a basis of $\ker(P, \mathcal{O}_{w_0}) \cong \mathbf{C}^3$. Then the monodromy defines the representation of the fundamental group

$$\pi_1(\mathbf{C} \setminus \{0, 1, \coth^2 \kappa\}) \to GL(3, \mathbf{C}).$$

Now, we consider the subgroup $F_2 = \pi_1(\Omega \setminus \{0,1\})$, which is isomorphic to the free group of two generators. Restricting the above representation to the subgroup F_2, we have a representation,

$$\rho : F_2 \to GL(3, \mathbf{C}),$$

which is called a *restricted* monodromy representation.

Proposition 22. *The set of invariants of the restricted monodromy representation ρ is isomorphic to the solutions*

$$\ker(P : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)) \xrightarrow{\sim} (\mathbf{C}^3)^{\rho(F_2)}.$$

We take two loops γ_0, $\gamma_1 \in \pi_1(\Omega \setminus \{0,1\})$ around 0, 1, respectively, and consider the corresponding local monodromy matrices $\rho(\gamma_0), \rho(\gamma_1) \in GL(3, \mathbf{C})$. Then the set of invariants is equivalent to the common kernel of these two matrices:

$$(\mathbf{C}^3)^{\rho(F_2)} = \{f \in \mathbf{C}^3 \mid \rho(\gamma_0)f = f, \rho(\gamma_1)f = f\}.$$

Example 23. *For a non-generic parameter ν, we have the following properties for local monodromy matrices:*

(i) The case $(2\nu - 1)/4 \in \mathbf{Z}_+$. *The matrix $\rho(\gamma_1)$ is semisimple with eigenvalues* $1, 1$ *and* -1, *while the matrix $\rho(\gamma_0)$ has eigenvalues* $1, 1$ *and* -1. *Then the dimension of the common kernel is* 0, 1, *or* 2.

(ii) The case $(2\nu + 1)/4 \in \mathbf{Z}_+$. *The matrix $\rho(\gamma_0)$ has eigenvalues* $1, -1$ *and* -1. *The eigenspace for the eigenvalue* 1 *is one-dimensional. The matrix $\rho(\gamma_1)$ has eigenvalues* $1, 1$ *and* 1. *The dimension of the common kernel is* 0 *or* 1.

(iii) The case of odd functions with $(2\nu+1)/4 \in \mathbf{Z}_+$. *We can define the restricted monodromy representation for the operator appearing in Section 4.3:*

$$\rho^o : F_2 \to GL(2, \mathbf{C}).$$

We have a relation similar to that appearing in Proposition 22,

$$\ker(H(w, D_w), \mathcal{O}(\Omega)) \xrightarrow{\sim} (\mathbf{C}^2)^{\rho^o(F_2)} = \{f \in \mathbf{C}^2 \mid \rho^o(\gamma_0)f = \rho^o(\gamma_1)f = f\}.$$

Both the matrices $\rho^o(\gamma_0)$ and $\rho^o(\gamma_1)$ are unipotent (with eigenvalues $1, 1$*). The dimension of the kernel is* 0, 1, *or* 2.

4.A. Positivity of eigenvalues. Assume equation (2.3) holds. Then for $u = (u_-, u_+)$,

$$\bar{u}_-\pi(S_-)u_- + \bar{u}_+\pi(S_+)u_+ = \mu\delta(|u_+|^2 + |u_-|^2 - \varepsilon^2(\bar{u}_+u_- + \bar{u}_-u_+).$$

The right-hand side of this expression is

$$\mu\delta((1-\varepsilon^2)(|u_+|^2 + |u_-|^2) + \varepsilon^2|u_+ + u_-|^2).$$

On the other hand,

$$\int \bar{u}_-\pi(S_-)u_- + \bar{u}_+\pi(S_+)u_+ = \frac{1}{2}\int(|\partial_x u_-|^2 + |xu_+|^2).$$

This proves the relations $\mu\delta > 0$ and $\nu = \mu\delta\cosh\kappa > 0$.

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